Relative Property (T) Actions and Trivial Outer Automorphism Groups

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Abstract

We show that every non-amenable free product of groups admits free ergodic probability measure preserving actions which have relative property (T) in the sense of S. Popa [Pop06, Def. 4.1]. There are uncountably many such actions up to orbit equivalence and von Neumann equivalence, and they may be chosen to be conjugate to any prescribed action when restricted to the free factors. We exhibit also, for every non-amenable free product of groups, free ergodic probability measure preserving actions whose associated equivalence relation has trivial outer automorphisms group. This gives in particular the first examples of such actions for the free group on 2 generators.

1 Introduction

Several breakthroughs in von Neumann Algebras theory and Orbit Equivalence have been made possible, during the last years, by the introduction and the study by S. Popa of the notion of **rigidity** for pairs $B \subset N$ of von Neumann algebras [Pop06]. This property, inspired by relative property (T) of Kazhdan, satisfied for the inclusion $L^\infty(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$ associated with the standard action of $\mathrm{SL}(2,\mathbb{Z})$ on the 2-torus, has proved to be extremely useful, more generally, for inclusions $A \subset M(\mathcal{R})$ arising (as Cartan subalgebra A embedded in the generalized crossed product or group-measure-space construction von Neumann algebra [MvN36], [FM77b]) from standard countable probability measure preserving (p.m.p.) equivalence relations $\mathcal{R}$. In case the pair $A \subset M(\mathcal{R})$ is rigid, the relation is said to have the **relative property (T)**. When combined with antagonistic properties like Haagerup property [Pop06] or (amalgamated) free product decomposition [IPP05], the rigid Cartan subalgebra were shown to be essentially unique, thus allowing orbit equivalence invariants (like the cost [Gab00], L^2-Betti numbers $\beta_n(\mathcal{R})$ [Gab02] or the fundamental group $\mathcal{F}(\mathcal{R})$ [Gab02, Cor. 5.7],...) to translate to von Neumann algebras invariants. This led to the solution of the long standing problem of finding von Neumann II_1 factors with trivial fundamental group [Pop06].

While the class of groups which admit a free p.m.p. ergodic relative property (T) action is closed under certain algebraic constructions (like direct products, commensurability [Pop06], or free product with an arbitrary group [IPP05, Cor. 7.15]), the building blocks were very few and relying on some arithmetic actions [Pop06], [Val05], [Fer06] leaving open the general problem [Pop06, Prob. 5.10.2.]: *"Characterize the countable discrete groups Γ_0 that can act rigidly on the probability space (X,μ), i.e., for which there exist free ergodic measure preserving actions σ on (X,μ) such that $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes_\sigma \Gamma_0$ is a rigid embedding."*

We prove that the class of such groups contains all non-amenable free products of groups. Moreover, we exhibit many different actions (Th. 1.2) and remove any arithmetic assumption on the individual actions of the factors, while in fact, given the state of art, an arithmetic flavor may remain hidden in the way the individual actions are mutually arranged.

Recall from [Pop06, Def. 4.1] the definition of a rigid inclusion (or of relative rigidity of a subalgebra).

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Definition 1.1 *Let M be a factor of type II_1 with normalized trace τ and let $A \subset M$ be a von Neumann subalgebra. The inclusion $A \subset M$ is called* ***rigid*** *if the following property holds: for every $\epsilon > 0$, there exists a finite subset $\mathcal{J} \subset M$ and a $\delta > 0$ such that whenever ${}_M H_M$ is a Hilbert M-M-bimodule admitting a unit vector ξ with the properties*

- $\|a \cdot \xi - \xi \cdot a\| < \delta$ *for all* $a \in \mathcal{J}$,
- $|\langle \xi, a \cdot \xi\rangle - \tau(a)| < \delta$ *and* $|\langle \xi, \xi \cdot a\rangle - \tau(a)| < \delta$ *for all a in the unit ball of M,*

there exists a vector $\xi_0 \in H$ satisfying $\|\xi - \xi_0\| < \epsilon$ and $a \cdot \xi_0 = \xi_0 \cdot a$ for all $a \in A$.

In the case of a relatively rigid Cartan subalgebra in a (generalized) crossed-product $L^\infty(X,\mu) \subset L^\infty(X,\mu) \ltimes_\sigma \Gamma$ or $L^\infty(X,\mu) \subset M(\mathcal{R})$, the free ergodic action $\Gamma \overset{\sigma}{\curvearrowright} (X,\mu)$ (resp. the standard equivalence relation $\mathcal{R}$) is called **rigid** or is said to have the **relative property (T)**.

Recall the following weaker and weaker notions of equivalence for p.m.p. actions or standard equivalence relations $\mathcal{R}, \mathcal{S}$ on (X,μ):
They are **Orbit Equivalent** if there is a p.m.p. isomorphism of the space that sends classes to classes

$$\mathcal{R} \overset{\mathrm{OE}}{\sim} \mathcal{S} \tag{1}$$

or equivalently if the associated pairs are isomorphic

$$\big(L^\infty(X,\mu) \subset M(\mathcal{R})\big) \quad \simeq \quad \big(L^\infty(X,\mu) \subset M(\mathcal{S})\big) \tag{2}$$

They are **von Neumann Equivalent** if solely the generalized crossed products are isomorphic

$$M(\mathcal{R}) \quad \simeq \quad M(\mathcal{S}) \tag{3}$$

They are **von Neumann Stably Equivalent** if the generalized crossed products are stably isomorphic for some $r \in \mathbb{R}_+^*$

$$M(\mathcal{R}) \quad \simeq \quad M(\mathcal{S})^r \tag{4}$$

We obtained [GP05] that the non-cyclic free groups admit uncountably many relative property (T) orbit inequivalent and even von Neumann stably inequivalent free ergodic actions. Observe, by [Dye59], that there is essentially a large liberty for the conjugacy classes of the restrictions of the actions to elements of a base. We extend this to any free product:

Theorem 1.2 *Every non-trivial ($|I| > 1$) free product of countable infinite groups*

$$\Gamma = \mathop{*}_{i \in I} \Gamma_i \tag{5}$$

admits continuum many von Neumann stably inequivalent relative property (T) free ergodic actions, whose restriction to each factor is conjugated with a prescribed action.

More precisely, let $\Gamma_i \overset{\sigma_i}{\curvearrowright} (X,\mu)$ be an at most countable collection of p.m.p. free actions of countable infinite groups Γ_i ($|I| \neq 1$) on the standard probability space. There exists continuum many von Neumann inequivalent free ergodic actions $(\alpha_t)_{t \in T}$ of the free product $\Gamma = *_{i\in I}\Gamma_i$ that have relative property (T), and such that for every $t \in T$ and $i \in I$, the restriction $\alpha_t|\Gamma_i$ of α_t to Γ_i is conjugated with σ_i

$$\big(\Gamma_i \overset{\alpha_t|\Gamma_i}{\curvearrowright} (X,\mu)\big) \overset{\mathrm{Conj}}{\sim} \big(\Gamma_i \overset{\sigma_i}{\curvearrowright} (X,\mu)\big) \tag{6}$$

Observe that A. Ioana [Ioa07, Th. 4.3] exhibited for every non-amenable group, actions σ satisfying a weak form of relative rigidity, namely for which there exists a diffuse $Q \subset L^\infty(X,\mu) \subset M(\mathcal{R}_\sigma)$ such that Q is relatively rigid in $M(\mathcal{R}_\sigma)$ and has relative commutant contained in $L^\infty(X,\mu)$.

We introduced in [Gab00] in connection with cost, the notion of freely independent equivalence relations and of free decomposition of an equivalence relation (see [Alv08] for a geometric approach):

$$\mathcal{S} = \mathop{*}_{i\in I} \mathcal{S}_i \tag{7}$$

The following, essentially due to A. Törnquist [Tör06], see also [IPP05, Prop. 7.3], states that equivalence relations may be put in general position:

Theorem 1.3 *Let $(\mathcal{S}_i)_{i\in I}$ be a countable collection of standard p.m.p. equivalence relations on (X,μ), then there exists an equivalence relation $\mathcal{S}$ on (X,μ) generated by a family of freely independent subrelations $\mathcal{S}'_i$ such that $\mathcal{S}'_i \overset{\mathrm{OE}}{\sim} \mathcal{S}_i$, $\forall i \in I$.*

We obtain in fact the following more general form of theorem 1.2:

Theorem 1.4 *Let $(\mathcal{S}_i)_{i\in I}$ be a countable collection of p.m.p. standard countable equivalence relations on the standard non atomic probability space (X,μ) whose classes are almost all infinite, $|I| \geq 2$. Then there exists continuum many von Neumann stably inequivalent relative property (T) ergodic p.m.p. equivalence relations on (X,μ) generated by a freely independent family of subrelations $\mathcal{S}'_i$ such that for every $i\in I$, $\mathcal{S}'_i \overset{\mathrm{OE}}{\sim} \mathcal{S}_i$.*
More precisely, there exists a strictly increasing continuum of ergodic equivalence relations $\mathcal{S}^t$, $t\in(0,1]$ such that

1. *for every $t\in(0,1]$, we have a free decomposition $\mathcal{S}^t = \mathop{*}_{i\in I} \mathcal{S}^t_i$ such that $\mathcal{S}^t_i \overset{\mathrm{OE}}{\sim} \mathcal{S}_i$ for all $i \in I$*
2. *for every $t\in(0,1]$, $\lim\limits_{s\to t, s<t} \nearrow \mathcal{S}^s = \mathcal{S}^t$*
3. *$A = L^\infty(X,\mu)$ is a relatively rigid Cartan subalgebra of the associated II_1 factor $M(\mathcal{S}^t)$*
4. *the classes of stable isomorphism among the $M(\mathcal{S}^t)$ are at most countable, in particular there is an uncountable set T such that for $t\in T\subset(0,1]$ the $M(\mathcal{S}^t)$ are pairwise not stably isomorphic.*

It follows, from [IPP05, Cor. 7.13] that, under mild conditions on L^2-Betti numbers (see [Gab02]), we thus produce plenty of new examples of factors with trivial fundamental group:

Corollary 1.5 *If the L^2-Betti numbers satisfy $\sum_{i\in I}\beta_n(\mathcal{S}_i) \notin \{0,\infty\}$ for some $n\neq 1$, or $|I| + \sum_{i\in I}\beta_1(\mathcal{S}_i) \neq \infty$ the above factors have trivial fundamental group: $\mathcal{F}(M(\mathcal{S}^t)) = \{1\}$.*

Outer automorphism groups of equivalence relations are usually hard to calculate and there are only few special families of group actions for which one knows that $\mathrm{Out}(\mathcal{R}) = \{1\}$. The first examples are due to S. Gefter [Gef93], [Gef96] and then A. Furman [Fur05]. Monod-Shalom [MS06] produced an uncountable family of non orbit-equivalent actions. They all take advantage of the setting of higher rank lattices. Using rigidity results from [MS06], Ioana-Peterson-Popa [IPP05] gave the first shift-type examples. However, all these examples are concerned with very special kind of actions. And the free groups keep out of reach by these technics.

The first general result appeared extremely recently in a paper of Popa-Vaes and concerns free products of any countable group Λ with the free group on infinitely many generators $\mathbf{F}_\infty$, with no condition at all on the Λ-action:

Theorem 1.6 ([PV08, Th. 4.3]) *Let $\Lambda \overset{\sigma}{\curvearrowright} (X,\mu)$ be any essentially free, probability measure preserving action of any countable group. There exists an uncountable family $(\sigma_t)_{t\in T}$ of essentially free, ergodic, probability measure preserving actions of $\mathbf{F}_\infty * \Lambda$ on (X,μ) with the following properties:*

- *The orbit equivalence relation of $\mathcal{R}$ has trivial fundamental group and trivial outer automorphism group.*
- *The restriction of σ_t to Λ is conjugate with σ.*
- *The actions $(\sigma_t)_{t\in T}$ are not stably orbit equivalent (and even not stably von Neumann equivalent)*

- *The restriction of σ_t to $\mathbf{F}_\infty$ is ergodic and relative property (T).*

Largely inspired by their technics, we extend this kind of results to the non-cyclic free groups on finitely many generators, and more generally to any free product without any condition on the building blocks.

Theorem 1.7 *There exists continuum many free actions $(\sigma_t)_{t\in T}$ of $\mathbf{F}_r$, $r = 2, 3, \cdots, \infty$ such that :*

- $\mathrm{Out}(\mathcal{R}_{\sigma_t}) = \{1\}$
- *the actions σ_t are pairwise von Neumann stably inequivalent.*
- *the actions σ_t are ergodic and relative property (T)*

This follows from the more general:

Theorem 1.8 *Let $(\mathcal{S}_i)_{i\in I}$, $|I| \geq 2$, be a countable collection of p.m.p. standard countable equivalence relations on the standard non atomic probability space (X, μ) whose classes are almost all infinite. Then there exists a continuum of equivalence relations $\mathcal{S}^t$ with the following properties:*

- $\mathrm{Out}(\mathcal{S}^t) = \{1\}$
- $\mathcal{S}^t = \underset{i\in I}{*} \mathcal{S}_i^t$ *for some* $\mathcal{S}_i^t \overset{\mathrm{OE}}{\sim} \mathcal{S}_i$ *for all $i \in I$ and for all $t \in T$*
- *the relations $(\mathcal{S}^t)_{t\in T}$ are von Neumann inequivalent*
- *the relations $(\mathcal{S}^t)_{t\in T}$ are ergodic and relative property (T)*

Corollary 1.9 *Every non-trivial free product of countable infinite groups $*_{i\in I}\Gamma_i$ admits continuum many von Neumann stably inequivalent relatively free ergodic $(\sigma_t)_{t\in T}$ actions with $\mathrm{Out}(\mathcal{R}_{\sigma_t}) = \{1\}$ and whose restriction to each factor is conjugated with a prescribed action.*

2 Proofs

Notation

- $\mathcal{R}|Y$ the restriction of $\mathcal{R}$ to a Borel subset $Y \subset X$.
- $\mathbf{F}_n$ the free group on n generators
- $\mathcal{R}_\beta$ the equivalence relation associated with an action β.
- $\langle \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_k\rangle$ the equivalence relation generated by $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_k$.
- $\langle \varphi_1, \varphi_2, \cdots, \varphi_n\rangle$ the equivalence relation generated by the family of partial isomorphisms $(\varphi_1, \varphi_2, \cdots, \varphi_n)$.
- $\mathrm{dom}(\varphi)$ and $\mathrm{rng}(\varphi)$ denote the domain and range of the partial isomorphism φ
- $\mathcal{R}_1 * \mathcal{R}_2$ means that $\mathcal{R}_1$, $\mathcal{R}_2$ are freely independent, and $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ that $\mathcal{R}$ is generated by the freely independent subrelations $\mathcal{R}_1$ and $\mathcal{R}_2$

2.1 Proof of Th. 1.4

The general result follows from the case $|I|=2$ by splitting $(\mathcal{S}_i)_{i\in I}$ into two families $(\mathcal{S}_i)_{i\in I'}$ and $(\mathcal{S}_i)_{i\in I''}$, and putting the $(\mathcal{S}_i)_{i\in I'}$ (resp. $(\mathcal{S}_i)_{i\in I''}$) mutually freely independent by Th. 1.3.

We will use in several points the following powerful basic observation. For further applications, we don't want to require the relations $\mathcal{S}_i$ to be ergodic:

Theorem 2.1 *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be two measure preserving countable standard equivalence relations on the standard non atomic probability space (X,μ), with almost only infinite classes.*
-1- For every integer $p\in\mathbb{N}\setminus\{0,1\}$, there are Y, Y_2 Borel subsets of X of measure $\frac{1}{p}$ and there is on Y a relation $\mathcal{U}_2$ and a free action σ of $\mathbf{F}_{p-1}$ such that $\mathcal{S}^\sigma:=\langle\mathcal{S}_1,\mathcal{U}_2,\mathcal{R}_\sigma\rangle$ satisfies

$$\mathcal{S}^\sigma = \mathcal{S}_1*\mathcal{U}_2*\mathcal{R}_\sigma=\mathcal{S}_1*\mathcal{S}_2^\sigma \quad \textit{for some} \quad \mathcal{S}_2^\sigma \overset{\mathrm{OE}}{\sim} \mathcal{S}_2 \tag{8}$$

$$\mathcal{S}^\sigma|Y = \mathcal{S}_1|Y*\mathcal{U}_2*\mathcal{R}_\sigma \tag{9}$$

-2- For any free $\mathbf{F}_{p-1}$-action β on Y such that $\mathcal{R}_\beta$ is freely independent from $\mathcal{S}_1|Y*\mathcal{U}_2$, the relation $\mathcal{S}^\beta:=\langle\mathcal{S}_1,\mathcal{U}_2,\mathcal{R}_\beta\rangle$ satisfies

$$\mathcal{S}^\beta = \mathcal{S}_1*\mathcal{U}_2*\mathcal{R}_\beta=\mathcal{S}_1*\mathcal{S}_2^\beta \quad \textit{for some} \quad \mathcal{S}_2^\beta \overset{\mathrm{OE}}{\sim} \mathcal{S}_2 \tag{10}$$

$$\mathcal{S}^\beta|Y = \mathcal{S}_1|Y*\mathcal{U}_2*\mathcal{R}_\beta \tag{11}$$

-3- The $\mathbf{F}_{p-1}$-actions β above may be required to be conjugated with any prescribed free action α of $\mathbf{F}_{p-1}$.
-4- If the $\mathbf{F}_{p-1}$-action β is relatively rigid then $A:=L^\infty(X)$ is embedded

$$A\subset M(\mathcal{S}^\beta) \tag{12}$$

as a relatively rigid Cartan subalgebra.

In case the $\mathcal{S}_i$ were ergodic, the point -1- follows immediately from Th. 1.3 and [IPP05, Prop.7.4 2°]. Consider for $i=1,2$ the $\mathcal{S}_i$-ergodic decomposition $m_i:(X,\mu)\to\mathcal{EM}_i$ (where $\mathcal{EM}_i$ is the space of $\mathcal{S}_i$-invariant ergodic measures) (see [Var63] and consider $\mathcal{S}_i$ as given by a – non-necessarily free –group action [FM77a, Th. 1]), and the Borel subsets $X_i(s,r):=\{x\in X_i : s\leq m_i(x)([0,x])\leq r\}$. This set meets each ergodic component along a subset of measure $r-s$: for each ergodic measure $e\in\mathcal{EM}_i$, $e(X_i(s,r))=e(X_i(s,r)\cap m_i^{-1}(e))=r-s$. The usual argument to find a partial isomorphism in the full groupoid of an ergodic relation between two subsets of equal measure extends immediately to deliver partial isomorphisms $\varphi_i^j: Y_i^0\to Y_i^j$ (for $j=1,\cdots,p-1$) in the full groupoid $[[\mathcal{S}_i]]$ where $Y_i^j:=X_i(\frac{j}{p},\frac{j+1}{p})$. Then

$$\mathcal{S}_i=\mathcal{S}_i|Y_i^0*\langle\varphi_i^1\rangle*\langle\varphi_i^2\rangle*\cdots\langle\varphi_i^{p-1}\rangle,\quad i=1,2 \tag{13}$$

By Th. 1.3, there is on $Y:=Y_1^0$ an equivalence relation $\mathcal{U}_2\overset{\mathrm{OE}}{\sim}\mathcal{S}_2|Y_2$, where $Y_2:=Y_2^0$, which is freely independent from $\mathcal{S}_1|Y$.

Let $a_1,a_2,\cdots,a_{p-1}$ be a free generating set of the free group $\mathbf{F}_{p-1}$ and β a free $\mathbf{F}_{p-1}$-action such that $\mathcal{R}_\beta$ is freely independent from $\mathcal{S}_1|Y*\mathcal{U}_2$. Since $\langle\varphi_1^1\rangle*\langle\varphi_1^2\rangle*\cdots\langle\varphi_1^{p-1}\rangle$ is smooth with fundamental domain Y, it is freely independent from any relation whose classes are singletons for every point $x\in X\setminus Y$, thus:

$$\begin{aligned}
\mathcal{S}^\beta &= \mathcal{S}_1*\mathcal{U}_2*\mathcal{R}_\beta &(14)\\
&= \mathcal{S}_1|Y*\langle\varphi_1^1\rangle*\langle\varphi_1^2\rangle*\cdots\langle\varphi_1^{p-1}\rangle*\mathcal{U}_2*\langle\beta(a_1)\rangle*\langle\beta(a_2)\rangle*\cdots*\langle\beta(a_{p-1})\rangle &(15)\\
&= \mathcal{S}_1|Y*\langle\varphi_1^1\rangle*\langle\varphi_1^2\rangle*\cdots\langle\varphi_1^{p-1}\rangle*\underbrace{\mathcal{U}_2*\langle\varphi_1^1\circ\beta(a_1)\rangle*\langle\varphi_1^2\circ\beta(a_2)\rangle*\cdots*\langle\varphi_1^{p-1}\circ\beta(a_{p-1})\rangle}_{\mathcal{S}_2^\beta} &(16)
\end{aligned}$$

The partial isomorphisms $\varphi_1^j\circ\beta(a_j)$ have domain Y and target Y_1^j, so that the composition $f^j=\varphi_2^j\circ f^0\circ(\varphi_1^j\circ\beta(a_j))^{-1}: Y_1^j\to Y_2^j$, where f^0 is the isomorphism $Y\to Y_2^0$ witnessing $\mathcal{U}_2\overset{\mathrm{OE}}{\sim}\mathcal{S}_2|Y_2$, fit together to give an OE

between $\mathcal{S}_2^\beta$ and $\mathcal{S}_2$. Again by Th. 1.3, any prescribed free $\mathbf{F}_{p-1}$-action may be conjugated to an action β that is freely independent from $\mathcal{S}_1|Y * \mathcal{U}_2$.

The pair (12) is obtained by amplifying the pair $(B \subset M(\mathcal{S}^\beta|Y))$ with $B := L^\infty(Y)$, which contains the pair

$$B \subset M(\mathcal{R}_\beta) \tag{17}$$

It then follows from [Pop06, Prop. 4.4 2°, 4.5.3°, 5.1, 5.2] that rigidity of the pair (17) entails rigidity of the pair (12). This concludes the proof of Theorem 2.1. □

Consider now a free $\mathbf{F}_{p-1}$-action β that is conjugated with a relative property (T) action α_1 such that one group element in some free generating set of $\mathbf{F}_{p-1}$ acts ergodically, for instance the standard free action on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ of some $\mathbf{F}_{p-1} \subset \mathrm{SL}(2,\mathbb{Z})$ (see [GP05]). In [GP05, Prop. 1] we constructed a continuum of actions α_t of free ergodic actions of $\mathbf{F}_{p-1}$, indexed by $t \in (0,1]$ defining a strictly increasing continuum of standard m.p. equivalence relation $\mathcal{Y}_t$ such that for each $t \in (0,1]$, $\lim_{s\to t, s<t} \nearrow \mathcal{Y}_s = \mathcal{Y}_t$. Since $\mathcal{Y}_t$ is a subrelation of $\mathcal{Y}_1$, it follows that $\mathcal{S}_1|Y, \mathcal{U}_2, \mathcal{Y}_t$ are freely independent and from theorem 2.1:

$$\mathcal{S}^t := \mathcal{S}_1 * \mathcal{U}_2 * \mathcal{Y}_t = \mathcal{S}_1 * \mathcal{S}_2^t \quad \text{for some} \quad \mathcal{S}_2^t \overset{\mathrm{OE}}{\sim} \mathcal{S}_2 \tag{18}$$

$$\mathcal{S}^t|Y = \mathcal{S}_1|Y * \mathcal{U}_2 * \mathcal{Y}_t \tag{19}$$

The family of factors $M_t = M(\mathcal{S}^t)$ associated with the equivalence relations $\mathcal{S}^t$ thus satisfies the hypothesis of [GP05, Prop. 4]: it is a strictly increasing family of subfactors $A \subset M_t \subset M_1$, with $A = L^\infty(X)$ Cartan, such that $M_1 = \overline{\cup_{t<1} M_t}$ and $A \subset M_1$ relatively rigid. Then, there is $c < 1$ such that for $t \in [c, 1)$,
(i) the pairs $A \subset M_t$ are relatively rigid Cartan subalgebras,
(ii) the classes of stable isomorphism among the M_t are at most countable, in particular there is an uncountable set J such that for $t \in J \subset (c, 1]$ the M_t are pairwise not stably isomorphic.
Observe that moreover
(iii) M_t has at most countable fundamental group (by [NPS07, Th. A.1])
(iv) M_t has at most countable outer automorphism group (by [Pop06, Th. 4.4]). The proof of th. 1.4 is complete. □

2.2 Proof of Theorem 1.8

It relies on the following:

Theorem 2.2 *Let $\mathcal{R}_0$ be an ergodic relative property (T) p.m.p. equivalence relation on the non atomic standard probability space (X, μ). There exist continuum many p.m.p. ergodic equivalence relations $(\mathcal{R}^t)_{t\in T}$, all contained in a standard equivalence relation $\overline{\mathcal{R}} = \mathcal{R}_0 * \mathcal{R}_\sigma$ for some free action of $\mathbf{F}_2$, such that*

1. *$\mathcal{R}^t = \mathcal{R}_0 * \langle g^t \rangle$ for some isomorphism $g^t : X \to X$*
2. *the $\mathcal{R}^t$ have relative property (T) and pairwise von Neumann inequivalent*
3. $\mathrm{Out}(\mathcal{R}^t) = \{1\}$

Assuming this, we prove Theorem 1.8: The general result follows from the case $|I| = 2$ by splitting $(\mathcal{S}_i)_{i\in I}$ into two families $(\mathcal{S}_i)_{i\in I'}$ and $(\mathcal{S}_i)_{i\in I''}$, and putting the $(\mathcal{S}_i)_{i\in I'}$ (resp. $(\mathcal{S}_i)_{i\in I''}$) mutually freely independent by theorem 1.3. Apply theorem 2.1 to $\mathcal{S}_1 * \mathcal{S}_2$ with $p = 4$, so as to obtain

$$\mathcal{S}^\sigma = \mathcal{S}_1 * \mathcal{U}_2 * \mathcal{R}_\sigma = \mathcal{S}_1 * \mathcal{S}_2^\sigma \quad \text{for some} \quad \mathcal{S}_2^\sigma \overset{\mathrm{OE}}{\sim} \mathcal{S}_2 \tag{20}$$

$$\mathcal{S}^\sigma|Y = \mathcal{S}_1|Y * \mathcal{U}_2 * \mathcal{R}_\sigma \tag{21}$$

and σ some free action of $\mathbf{F}_3 = \langle a, b, c \rangle$. Then further decompose $\mathcal{R}_\sigma = \mathcal{R}_{\sigma\langle a,b\rangle} * \mathcal{R}_{\sigma\langle c\rangle}$ and consider any relative property (T) ergodic free action α of $\mathbf{F}_2 = \langle a, b \rangle$. Now, applying theorem 2.2 to the equivalence relation

$\mathcal{R}_0 := \mathcal{S}_1|Y * \mathcal{U}_2 * \mathcal{R}_\alpha$ on Y, we get the family $\mathcal{R}^t = \mathcal{R}_0 * \langle g^t \rangle = \mathcal{S}_1|Y * \mathcal{U}_2 * \mathcal{R}_\alpha * \langle g^t \rangle$ for some isomorphism $g^t : X \to X$. Now, $\mathcal{R}_\alpha * \langle g^t \rangle$ may be seen as a produced by a free action β^t of $\mathbf{F}_3$ to which theorem 2.1 -2- applies:

$$\mathcal{S}^{\beta^t} := \langle \mathcal{S}_1, \mathcal{U}_2, \mathcal{R}_{\beta^t} \rangle = \mathcal{S}_1 * \mathcal{U}_2 * \mathcal{R}_{\beta^t} = \mathcal{S}_1 * \mathcal{S}_2^{\beta^t} \quad \text{for some} \quad \mathcal{S}_2^{\beta^t} \overset{\mathrm{OE}}{\sim} \mathcal{S}_2 \tag{22}$$

$$\mathcal{S}^{\beta^t}|Y = \mathcal{S}_1|Y * \mathcal{U}_2 * \mathcal{R}_\alpha * \langle g^t \rangle = \mathcal{S}_1|Y * \mathcal{U}_2 * \mathcal{R}_{\beta^t} \tag{23}$$

The properties 2. and 3. in theorem 2.2 being invariant under stable orbit equivalence of fixed amplification constant $(\mathcal{S}^{\beta^t}|Y)^4 = \mathcal{S}^{\beta^t}$, it follows that

1. $\mathcal{S}^t := \mathcal{S}^{\beta^t}$ have relative property (T) and pairwise von Neumann inequivalent
2. $\mathrm{Out}(\mathcal{S}^t) = \{1\}$

Theorem 1.8 is proved. □

2.3 Proof of Theorem 2.2

Lemma 2.3 *Let $\mathcal{R}$ be a p.m.p. equivalence relation on the standard non-atomic probability space (X, μ) and $\mathcal{R}_0 \subset \mathcal{R}$ an ergodic standard subrelation with relative property (T). If $(\Delta_t : X \to X)_{t \in T}$ is an uncountable family of (measure preserving) automorphisms such that $\forall t \in T$, $\Delta_t(\mathcal{R}_0) \subset \mathcal{R}$, then there exist $k \neq l \in T$ such that $\Delta_k = \Delta_l$ on a non-negligible Borel subset.*

The proof comes out from the arguments in [PV08, Th. 4.1, step 1]. The Δ_t induce trace-preserving isomorphisms $\big(A \subset M(\mathcal{R}_0)\big) \simeq \big(A \subset M(\Delta_t(\mathcal{R}_0))\big)$ leading to embeddings $\theta_t : M(\mathcal{R}_0) \subset M(\mathcal{R})$ with $\Delta_t(a) = \Delta_t^{-1} \circ a$ for all $a \in A = L^\infty(X, \mu)$. The Hilbert space $\mathcal{H} := L^2(M, \tau)$ with its standard $M - M$-bimodule structure inherits for each $i, j \in T$ an $M(\mathcal{R}_0) - M(\mathcal{R}_0)$-bimodule structure $\mathcal{H}_{i,j}$, given by

$$u \underset{i}{\cdot} \xi \underset{j}{\cdot} v = \Theta_i(u) \xi \Theta_j(v)$$

By separability of $\mathcal{H}$ and uncountability of T, the (tracial) vector ξ, image of Id in the standard embedding $M(\mathcal{R}) \subset L^2(M(\mathcal{R}), \tau)$ satisfies for $\epsilon < 1/2$ the condition of rigidity def. 1.1 for $A \subset M(\mathcal{R}_0)$, for some $j \neq k$. Thus there exists $\xi_0 \in L^2(M(\mathcal{R}), \tau)$ with

$$\|\xi_0 - \xi\| < \epsilon \tag{24}$$

such that $\forall a \in A$, $a \underset{k}{\cdot} \chi_0 = \chi_0 \underset{l}{\cdot} a$, i.e. $\Theta_k(a)\chi_0 = \chi_0 \Theta_l(a)$. By $A - A$-bimodularity of the projection $P : L^2(M(\mathcal{R}), \tau) \to L^2(A)$, its image $h := P(\xi_0) \in L^2(X, \mu)$ satisfies the same identities: $\forall a \in A = L^\infty(X)$, $(a \circ \Delta_k^{-1})h = (a \circ \Delta_l^{-1})h$. In particular, $\Delta_k = \Delta_l$ on the support of h ($h \neq 0$ by (24)). □

We construct two treeings $\Phi = (\varphi_n)_{n \in \mathbb{N} \setminus \{0\}}$ and $\Psi = (\psi_n)_{n \in \mathbb{N} \setminus \{0\}}$ such that
-a- $\mathrm{dom}(\psi_n) = \mathrm{dom}(\varphi_n)$, $\mathrm{rng}(\psi_n) = \mathrm{rng}(\varphi_n)$, and the families of $\mathrm{dom}(\psi_n)$ and of $\mathrm{rng}(\psi_n)$ both form a partition of X.
-b- $\overline{\mathcal{R}} := \langle \mathcal{R}_0, \Phi, \Psi \rangle = \mathcal{R}_0 * \langle \Phi \rangle * \langle \Psi \rangle$
and satisfying some additional properties.

For every $E \subset \mathbb{N} \setminus \{0\}$, let's denote $\Phi_E := \{\varphi_n : n \in E\}$ and $\Psi_{\overline{E}} = \{\psi_n : n \notin E\}$. The relations

$$\mathcal{R}_E := \mathcal{R}_0 * \langle \Phi_E \rangle \tag{25}$$

$$\widetilde{\mathcal{R}}_E := \mathcal{R}_0 * \langle \Phi_E \rangle * \langle \Psi_{\overline{E}} \rangle \tag{26}$$

will have relative property (T) since $\mathcal{R}_0$ is. By [Pop06, Th. 4.4] $\mathrm{Out}(\widetilde{\mathcal{R}}_E)$ is countable modulo the full group $[\widetilde{\mathcal{R}}_E]$.

• Observe that $\langle\Phi_E\rangle * \langle\Psi_{\overline{E}}\rangle$ is a treed equivalence relation and that the partial isomorphisms $\Phi_E \cup \Psi_{\overline{E}}$ fit together to form a single automorphism $g_E : X \to X$.

$$\widetilde{\mathcal{R}}_E = \mathcal{R}_0 * \langle g_E\rangle \tag{27}$$

• Construct the ψ_n by restricting a single automorphism of X which is freely independent from $\mathcal{R}_0$ (Th. 1.3) to disjoint pieces of measures $\epsilon_n > 0$ such that $\sum_{n\in\mathbb{N}\setminus\{0\}} \epsilon_n = 1$.

• The φ_n's are constructed inductively: For $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the auxiliary p.m.p. standard equivalence relation generated by $\mathcal{R}_0$, $\Phi_{\{1,2,\cdots,n\}}$, Ψ and all of the $\mathrm{Aut}(\widetilde{\mathcal{R}}_F)$ for $F \subset \{1, 2, \cdots, n\}$. The countability of $\mathrm{Aut}(\widetilde{\mathcal{R}}_F)$ modulo $[\widetilde{\mathcal{R}}_F]$ ensures that $\mathcal{S}_n$ is actually countable. For $n = 0$, simply define $\mathcal{S}_0$ to be generated by $\mathcal{R}_0$, Ψ and $\mathrm{Aut}(\mathcal{R}_0)$.

Choose an isomorphism g_n of X that is freely independent of $\mathcal{S}_n$, restrict it to $\mathrm{dom}(\psi_n)$ and, by ergodicity of $\mathcal{R}_0$, compose it by a certain $h \in [\mathcal{R}_0]$ such that $h \circ g_n(\mathrm{dom}(\psi_n)) = \mathrm{rng}(\psi_n)$. Then set $\varphi_n = h \circ g_n|\mathrm{dom}(\psi_n)$.

Theorem 2.4 *There is an uncountable family $\mathcal{E}$ of subsets of $\mathbb{N} \setminus \{0\}$ such that for all $E \in \mathcal{E}$, $\mathrm{Out}(\widetilde{\mathcal{R}}_E)$ is trivial for $E \in \mathcal{E}$ and the classes of von Neumann equivalence are at most countable.*

The proof follows precisely the lines of [PV08, Th. 4.1]. Choose an uncountable family $\mathcal{E}_1$ of infinite subsets of $\mathbb{N} \setminus \{0\}$ such that $E \cap F$ is finite for all $E \neq F$ in $\mathcal{E}_1$ and fix a $\Delta_E \in \mathrm{Aut}(\widetilde{\mathcal{R}}_E)$ for each $E \in \mathcal{E}_1$. We will show that at least one Δ_E belongs to $[\widetilde{\mathcal{R}}_E]$. It follows that the sub-family of those $\widetilde{\mathcal{R}}_E$ with non-trivial outer automorphisms group is at most countable.

Step 1. There exists $E, F \in \mathcal{E}_1$ such that $E \neq F$ and $\Delta_E(x) = \Delta_F(x)$ for all x in a non-negligible subset $U_{E,F} \subset X$. We apply lemma 2.3 to the family of Δ_E with $\Delta_E(\mathcal{R}_0) \subset \mathcal{R}_E \subset \widetilde{\mathcal{R}}$.

Step 2. Since $\mathcal{R}_0$ is ergodic and contained in $\widetilde{\mathcal{R}}_E \cap \widetilde{\mathcal{R}}_F$, there are $f_E \in [\widetilde{\mathcal{R}}_E]$ and $f_F \in [\widetilde{\mathcal{R}}_F]$ such that μ-almost everywhere in X

$$f_E\Delta_E = f_F\Delta_F \tag{28}$$

Step 3. Δ_E belongs to $[\widetilde{\mathcal{R}}_E]$: From the relation (eq. 28)

$$\Delta := f_E\Delta_E = f_F\Delta_F \quad \in \quad \mathrm{Aut}(\widetilde{\mathcal{R}}_E) \cap \mathrm{Aut}(\widetilde{\mathcal{R}}_F) \quad \subset \quad \mathrm{Aut}(\widetilde{\mathcal{R}}_{E\cap F}) \tag{29}$$

Let $n \in E$ be such that $n > \max(E \cap F)$. Since $\Delta \in \mathrm{Aut}(\widetilde{\mathcal{R}}_E)$, there is a partial isomorphism $h \in [[\widetilde{\mathcal{R}}_E]]$, and a non-negligible subset $U \subset X$ on which

$$\Delta\varphi_n = h\Delta \tag{30}$$

Let p be the smallest index for which $h \in [[\mathcal{R}_0 * \langle\Phi_{E\cap\{1,2,\cdots,p\}}\rangle * \langle\Psi_{\overline{E}}\rangle]]$. By the condition that the φ_p are freely independent from $\mathcal{S}_q$, for $q < p$, and since $\Delta \in \mathrm{Aut}(\widetilde{\mathcal{R}}_{E\cap F}) \subset [\mathcal{S}_{\max(E\cap F)}]$, the relation (eq. 30) entails that $p = n$. Again by independence, by isolating the letters $\varphi_n^{\pm 1}$ in the reduced expression of h, the relation (eq. 30) delivers a subword $w \in [[\mathcal{R}_0 * \langle\Phi_{E\cap\{1,2,\cdots,p-1\}}\rangle * \langle\Psi_{\overline{E}}\rangle]]$ such that $\Delta = w$ on some non negligible set. It follows by ergodicity of $\mathcal{R}_0$ that Δ and thus that Δ_E belongs to $[\widetilde{\mathcal{R}}_E]$.

We show that given $E \in \mathcal{E}_1$, there are most countably many $F \in \mathcal{E}_1$ such that $\widetilde{\mathcal{R}}_E \overset{\mathrm{OE}}{\sim} \widetilde{\mathcal{R}}_F$. Apply lemma 2.3 to an uncountable family of isomorphisms $\Delta_{F,E} : \widetilde{\mathcal{R}}_E \overset{\mathrm{OE}}{\sim} \widetilde{\mathcal{R}}_F$. Then at least two of them, for $F, F' \in \mathcal{E}_1$, coincide on some non-negligible Borel subset. Like in step 2 above, it follows by ergodicity of $\mathcal{R}_0 \subset \widetilde{\mathcal{R}}_F \cap \widetilde{\mathcal{R}}_{F'}$ that there are $f_F \in [\widetilde{\mathcal{R}}_F]$ and $f_{F'} \in [\widetilde{\mathcal{R}}_{F'}]$ such that μ-almost everywhere in X

$$f_F\Delta_{F,E} = f_{F'}\Delta_{F',E} \tag{31}$$

It follows that $\widetilde{\mathcal{R}}_F = \widetilde{\mathcal{R}}_{F'}$ and thus $F = F'$.

The $A \subset M(\widetilde{\mathcal{R}}_E)$ are all rigid. An isomorphism $\Theta : M(\widetilde{\mathcal{R}}_E) \simeq M(\widetilde{\mathcal{R}}_F)$ delivers the relatively rigid Cartan subalgebra $\Theta(A) \subset M(\widetilde{\mathcal{R}}_F) = M(\mathcal{R}_0) \underset{A}{*} M(\langle g_E\rangle)$. By [IPP05, 7.12], there is a unitary $u \in M(\widetilde{\mathcal{R}}_F)$ such that $u\Theta(A)u^* = A$, i.e. $u\Theta(.)u^*$ induces an orbit equivalence $\widetilde{\mathcal{R}}_E \overset{\mathrm{OE}}{\sim} \widetilde{\mathcal{R}}_F$. This completes the proof of theorem 2.2.
□

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